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Estimates for the first and second Bohr radii of Reinhardt domains

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Abstract

We obtain general lower and upper estimates for the first and the second Bohr radii of bounded complete Reinhardt domains in $\mathbb{C}^n$.

Keywords: Bohr radii; Several complex variables; Power series; Polynomials; Banach spaces; Unconditional basis

1. Introduction

A bounded complete Reinhardt domain R in $\mathbb{C}^n$ is a bounded complete n-circled domain, i.e., if $z=(z_1,\dots,z_n)\in R$, then $\lambda(e^{\theta_1 i}z_1,\dots,e^{\theta_n i}z_n)\in R$ for all $\lambda\in\mathbb{C}$, $|\lambda|\leqslant 1$ and all $\theta_1,\dots,\theta_n\in\mathbb{R}$. As usual, if $z\in\mathbb{C}^n$ and $\alpha\in(\mathbb{N}\cup\{0\})^n$, then we write $z^\alpha := z_1^{\alpha_1}\dots z_n^{\alpha_n}$, $\alpha! = \alpha_1!\dots\alpha_n!$ and $|\alpha| = \alpha_1+\dots+\alpha_n$. $\mathcal{P}(^m\mathbb{C}^n)$ is the space of all m-homogeneous complex valued polynomials $\sum_{|\alpha|=m} c_\alpha z^\alpha$. For $1\leqslant p\leqslant\infty$ we write, as usual, ℓ_p^n for $\mathbb{C}^n$ endowed with the norm $||z||_p = (\sum_{k=1}^n |z_k|^p)^{1/p}$, and $B_{\ell_p^n}$ for its open unit ball.

In the last years (see [1–10,14,15,17 p. 321–322,18–20,22]) a lot of attention has been given to several multidimensional generalizations of a classical theorem of Bohr

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[11], namely: *let* $f = \sum_{k=0}^{\infty} a_k z^k$ *be a holomorphic function on* $\mathbb{D}$, *the open unit disk of* $\mathbb{C}$, *such that* $|f(z)| \leqslant 1$ *for each* $z \in \mathbb{D}$. *Then* $\sum_{k=0}^{\infty} |a_k z^k| \leqslant 1$ *when* $|z| < \frac{1}{3}$, *and moreover the radius* $\frac{1}{3}$ *is the best possible.* This $\frac{1}{3}$ is called the Bohr radius of $\mathbb{D}$.

The *first Bohr radius* of a bounded complete Reinhardt domain R was defined by Boas and Khavinson [10] to be $K(R) := \sup r$, the supremum taken over all $0 \leqslant r \leqslant 1$ such that whenever the power series $\sum_{\alpha \in (\mathbb{N} \cup \{0\})^n} c_\alpha z^\alpha$ satisfies $|\sum_{\alpha \in (\mathbb{N} \cup \{0\})^n} c_\alpha z^\alpha| \leqslant 1$ for all $z \in R$, it follows that $\sum_{\alpha \in (\mathbb{N} \cup \{0\})^n} |c_\alpha z^\alpha| \leqslant 1$ for all $z \in rR$.

In [16, (1.3)], Dineen and Timoney, while investigating the existence of absolute monomial basis for spaces of holomorphic functions over infinite dimensional locally convex spaces, required a several variables version of the classical result of Bohr quoted above. They obtained a result, which in the terminology of Boas and Khavinson is an upper bound for the Bohr radius of the polydisc, that is for all $\varepsilon > 0$ there exists $C(\varepsilon) > 0$ such that $K(B_{\ell_\infty^n}) \leqslant C(\varepsilon) n^{\frac{1}{2}+\varepsilon}$ (see [10,15, 4.6, p. 321] for a clarification). In [10, Theorem 2], Boas and Khavinson obtained lower and upper bounds for $K(B_{\ell_\infty^n})$, and also the optimal lower estimate which holds for all complete Reinhardt domains [10, Theorem 3]. Then Aizenberg [1, Theorem 9] got lower and upper bounds for $B_{\ell_1^n}$, and finally Boas [9, Theorem 3] solved the problem for $B_{\ell_p^n}$, $(1 < p < \infty)$ completely. All these results are summarized in: for every choice of coefficients b_α and c_α

$$\begin{aligned} &\frac{1}{3\sqrt[3]{e}} \frac{1}{n^{1-\frac{1}{p}}} \leqslant K(B_{\ell_p^n}) < 3 \left(\frac{\log n}{n} \right)^{1-\frac{1}{p}} \quad \text{if } 1 \leqslant p \leqslant 2, \\ &\frac{1}{3} \frac{1}{\sqrt{n}} \leqslant K(B_{\ell_p^n}) < 2 \sqrt{\log n} \frac{1}{\sqrt{n}} \quad \text{if } 2 \leqslant p \leqslant \infty. \end{aligned} \tag{1.1}$$

Linking Bohr radii with local Banach space theory, a general study of the behavior of $K(B_X)$ with respect to the dimension of X, where $X = (\mathbb{C}^n, \|.\|)$ is a Banach space such that the canonical basis is normalized and 1-unconditional, is given in [12]. Our main estimates are based on a probabilistic tool from [12, Theorem 3.1]. This result was improved in [13, Theorem 3.1], and we are going to apply this version in Section 1 of this article to obtain lower and upper estimates of $K(R)$ for any bounded complete Reinhardt domain of $\mathbb{C}^n$.

In [1], Aizenberg introduced the *second Bohr radius* $B(R)$ of a bounded complete Reinhardt domain R as the largest $0 \leqslant r \leqslant 1$ such that whenever the power series $\sum_{\alpha \in (\mathbb{N} \cup \{0\})^n} c_\alpha z^\alpha$ satisfies $|\sum_{\alpha \in (\mathbb{N} \cup \{0\})^n} c_\alpha z^\alpha| \leqslant 1$ for all $z \in R$, it follows that $\sum_{\alpha \in (\mathbb{N} \cup \{0\})^n} \sup_{z \in rR} |c_\alpha z^\alpha| \leqslant 1$. As Aizenberg points out in [1], $B(B_{\ell_\infty^n}) = K(B_{\ell_\infty^n})$, and in particular $B(\mathbb{D}) = \frac{1}{3}$. Moreover in [1, Theorem 4], Aizenberg proved that

$$\frac{1}{3n} < 1 - \left(\frac{2}{3} \right)^{\frac{1}{n}} \leqslant B(R) \tag{1.2}$$

for any bounded complete Reinhardt domain R in $\mathbb{C}^n$ and $n \geqslant 2$.

In [9, Theorem 5], Boas proved that

$$\frac{1}{3n} < 1 - \left(\frac{2}{3}\right)^{\frac{1}{n}} \leqslant B(B_{\ell_p^n}) < 4\frac{\log n}{n} \quad \text{if } 1 \leqslant p \leqslant 2,$$
$$\frac{1}{3}\left(\frac{1}{n}\right)^{\frac{1}{2}+\frac{1}{p}} \leqslant B(B_{\ell_p^n}) < 4\left(\frac{\log n}{n}\right)^{\frac{1}{2}+\frac{1}{p}} \quad \text{if } 2 \leqslant p \leqslant \infty. \tag{1.3}$$

In Section 2 of our paper, we are going to obtain general estimates (that can be actually computed in many cases) for the second Bohr radius of bounded complete Reinhardt domains of $\mathbb{C}^n$.

2. General estimates for the first Bohr radius

Clearly, $K(R)$, the first Bohr radius of R, is the supremum over all $0 \leqslant r \leqslant 1$ such that

$$\sup_{z\in R} \sum_{\alpha\in(\mathbb{N}\cup\{0\})^n} |c_\alpha (rz)^\alpha| \leqslant \sup_{z\in R}\left| \sum_{\alpha\in(\mathbb{N}\cup\{0\})^n} c_\alpha z^\alpha\right| \tag{2.1}$$

for every power series $\sum_\alpha c_\alpha z^\alpha$ convergent on R. Given $m \in \mathbb{N}$ we define

$$K_m(R) := \sup r \in [0,1],$$

the supremum taken over all $0 \leqslant r \leqslant 1$ such that whenever the m-homogeneous polynomial $\sum_{|\alpha|=m} c_\alpha z^\alpha$ satisfies $|\sum_{|\alpha|=m} c_\alpha z^\alpha| \leqslant 1$ for all $z \in R$, then $\sum_{|\alpha|=m} |c_\alpha z^\alpha| \leqslant 1$ for all $z \in rR$. Obviously, $K_m(R) := \sup r \in]0,1]$ such that

$$\sup_{z\in R} \sum_{|\alpha|=m} |c_\alpha z^\alpha| \leqslant \frac{1}{r^m} \sup_{z\in R}\left|\sum_{|\alpha|=m} c_\alpha z^\alpha\right|; \tag{2.2}$$

moreover, $K(R) \leqslant K_m(R)$. For the following lemma note that for any bounded complete Reinhardt domain R in $\mathbb{C}^n$

$$||P||_R := \sup\{|P(x)| : x \in R\}$$

defines a norm on the space $\mathcal{P}(^m\mathbb{C}^n)$.

Lemma 2.1. *If R is a bounded complete Reinhardt domain in $\mathbb{C}^n$, then for each $m \in \mathbb{N}$*

$$\frac{1}{K_m(R)^m} = \sup\left\{\left|\sum_{|\alpha|=m} \varepsilon_\alpha c_\alpha z^\alpha\right| : z \in R, \left|\left|\sum_{|\alpha|=m} c_\alpha z^\alpha\right|\right|_R \leqslant 1, |\varepsilon_\alpha| \leqslant 1, |\alpha| = m\right\}. \tag{2.3}$$

Proof. Clearly,

$$\sup_{z\in R, |\varepsilon_\alpha|\leqslant 1} \left| \sum_{|\alpha|=m} \varepsilon_\alpha c_\alpha z^\alpha \right| = \sup_{z\in R} \sum_{|\alpha|=m} |c_\alpha z^\alpha|$$

for all m-homogeneous polynomials $\sum_{|\alpha|=m} c_\alpha z^\alpha$. To finish the proof it is enough to deduce from (2.2) that

$$\frac{1}{K_m(R)^m} = \sup\left\{ \sum_{|\alpha|=m} |c_\alpha z^\alpha| : z\in R, \left\| \sum_{|\alpha|=m} c_\alpha z^\alpha \right\|_R \leqslant 1 \right\},$$

which obviously gives the desired equality. □

By normalizing and applying Lemma 2.1 we get for every choice of coefficients b_α and c_α

$$\sup_{z\in R} \sum_{|\alpha|=m} |b_\alpha c_\alpha z^\alpha| \leqslant \frac{1}{K_m(R)^m} \sup_\alpha |b_\alpha| \sup_{z\in R} \left| \sum_{|\alpha|=m} c_\alpha z^\alpha \right|. \tag{2.4}$$

The next result links the first Bohr radius $K(R)$ and the sequence $(K_m(R))_{m=1}^\infty$.

Proposition 2.2. *Let R be a bounded complete Reinhardt domain in $\mathbb{C}^n$. Then we have*

$$\tfrac{1}{3} \inf_m K_m(R) \leqslant K(R) \leqslant \min\left\{ \tfrac{1}{3}, \inf_m K_m(R) \right\}.$$

The proof of this proposition is very similar to [12, Proof of Theorem 2.2] taking into account that the right part of equality (2.3) is, by definition, $\chi_M(\mathcal{P}(^m\mathbb{C}^n), ||.||_R)$, the unconditional basis constant of the monomials in $\mathcal{P}(^m\mathbb{C}^n)$ when endowed with the norm $||.||_R$, i.e., we have that for each $m\in\mathbb{N}$

$$\frac{1}{K_m(R)^m} = \chi_M(\mathcal{P}(^m\mathbb{C}^n), ||.||_R).$$

We will need the following probabilistic estimate from [13, Corollary 3.2] (see also [12, Theorem 3.1]).

Theorem 2.3. *Let $(\varepsilon_\alpha)_{|\alpha|=m}$ be a family of independent standard Bernoulli random variables on a probability space (Ω, μ) (each $\varepsilon : \Omega \to \{-1, 1\}$ takes the values $+1$ and -1 with equal probability $\frac{1}{2}$), and let c_α, $|\alpha| = m$, be scalars. There exists a constant*

$0 < C_m \leqslant 2^{\frac{3m-1}{2}} m^{\frac{3}{2}}$, *such that for each bounded circled set U in $\mathbb{C}^n$ we have*

$$\int_{\Omega} \sup_{z \in U} \left| \sum_{|\alpha|=m} c_\alpha \varepsilon_\alpha z^\alpha \right| d\mu$$

$$\leqslant \sqrt{\log n}\, C_m \sup_{|\alpha|=m} \left\{ |c_\alpha| \sqrt{\frac{\alpha!}{m!}} \right\} \sup_{z \in U} \left(\sum_{k=1}^n |z_k|^2 \right)^{\frac{m-1}{2}} \sup_{z \in U} \sum_{k=1}^n |z_k|.$$

As a consequence, we get the following upper estimate for $K_m(R)$.

Corollary 2.4. *If R is a bounded complete Reinhardt domain in $\mathbb{C}^n$, then for each m*

$$K_m(R)^m \leqslant D_m \sqrt{\log n} \left(\frac{\sup_{z \in R} (\sum_{k=1}^n |z_k|^2)^{1/2}}{\sup_{z \in R} \sum_{k=1}^n |z_k|} \right)^{m-1},$$

where $0 < D_m \leqslant \sqrt{m!}\, 2^{\frac{3m-1}{2}} m^{\frac{3}{2}}$.

Proof. As R is n-circled and balanced we have that

$$\left(\sup_{z \in R} \sum_{k=1}^n |z_k| \right)^m = \sup_{z \in R} \left| \sum_{k=1}^n z_k \right|^m = \sup_{z \in R} \left| \left(\sum_{k=1}^n z_k \right)^m \right| = \sup_{z \in R} \left| \sum_{|\alpha|=m} \frac{m!}{\alpha!} z^\alpha \right|.$$

Hence, by (2.4) we have for each $\omega \in \Omega$

$$\left(\sup_{z \in R} \sum_{k=1}^n |z_k| \right)^m = \sup_{z \in R} \left| \sum_{|\alpha|=m} \frac{m!}{\alpha!} \varepsilon_\alpha(\omega) \varepsilon_\alpha(\omega) z^\alpha \right|$$

$$\leqslant \frac{\sqrt{m!}}{K_m(R)^m} \sup_{z \in R} \left| \sum_{|\alpha|=m} \sqrt{\frac{m!}{\alpha!}} \varepsilon_\alpha(\omega) z^\alpha \right|.$$

By integrating this inequality and applying Theorem 2.3 we obtain

$$\left(\sup_{z \in R} \sum_{k=1}^n |z_k| \right)^m \leqslant \int_{\Omega} \frac{\sqrt{m!}}{K_m(R)^m} \sup_{z \in R} \left| \sum_{|\alpha|=m} \sqrt{\frac{m!}{\alpha!}} \varepsilon_\alpha(\omega) z^\alpha \right| d\omega$$

$$\leqslant \frac{\sqrt{m!}}{K_m(R)^m} \sqrt{\log n}\, C_m \sup_{z \in R} \left(\sum_{k=1}^n |z_k|^2 \right)^{\frac{m-1}{2}} \sup_{z \in R} \sum_{k=1}^n |z_k|$$

from where the conclusion follows. □

We need to introduce a new notation in order to be able to compare Bohr radii. Let R_1 and R_2 be two bounded complete Reinhardt domains in $\mathbb{C}^n$; define

$$S(R_1, R_2) := \inf\{b > 0 : R_1 \subset b R_2\}.$$

As an example, if R is a bounded complete Reinhardt domain in $\mathbb{C}^n$, then

$$S(R, B_{\ell_p^n}) = \sup_{z\in R}\left(\sum_{k=1}^{n} |z_k|^p\right)^{1/p}, \quad 1\leqslant p<\infty,$$
$$S(R, B_{\ell_\infty^n}) = \sup_{z\in R}\ \sup_{k=1,\dots,n} |z_k|. \tag{2.5}$$

In particular, it follows from Hölder's inequality that

$$S(B_{\ell_\infty^n}, B_{\ell_p^n}) = n^{\frac{1}{p}}, \quad 1\leqslant p\leqslant \infty,$$
$$S(B_{\ell_q^n}, B_{\ell_2^n}) = n^{\frac{1}{2}-\frac{1}{q}}, \quad 2\leqslant q\leqslant \infty. \tag{2.6}$$

Lemma 2.5. *For two bounded complete Reinhardt domains R_1 and R_2 in $\mathbb{C}^n$ we have*

$$\frac{1}{S(R_1,R_2)S(R_2,R_1)}K(R_2)\leqslant K(R_1)\leqslant S(R_1,R_2)S(R_2,R_1)K(R_2).$$

Proof. To check the first inequality, assume that

$$\left|\sum_\alpha c_\alpha z^\alpha\right|\leqslant 1 \quad \text{for all } z\in R_1.$$

Then, for every $\delta_1>0$

$$\left|\sum_\alpha c_\alpha\left(\frac{z}{S(R_2,R_1)+\delta_1}\right)^\alpha\right|\leqslant 1 \quad \text{for all } z\in R_2.$$

Hence, for every $0<\varepsilon<K(R_2)$,

$$\sum_\alpha |c_\alpha|\frac{1}{(S(R_2,R_1)+\delta_1)^{|\alpha|}}|z^\alpha|\leqslant 1 \quad \text{for all } z\in (K(R_2)-\varepsilon)R_2.$$

But this obviously implies that for every $\delta_2>0$

$$\sum_\alpha |c_\alpha|\frac{1}{(S(R_2,R_1)+\delta_1)^{|\alpha|}}\left|\left(\frac{K(R_2)-\varepsilon}{S(R_1,R_2)+\delta_2}z\right)^\alpha\right|$$
$$= \sum_\alpha |c_\alpha|\left|\left(\frac{K(R_2)-\varepsilon}{(S(R_2,R_1)+\delta_1)(S(R_1,R_2)+\delta_2)}z\right)^\alpha\right|\leqslant 1 \quad \text{for all } z\in R_1,$$

which finally shows

$$\frac{1}{S(R_1,R_2)S(R_2,R_1)}K(R_2)\leqslant K(R_1).$$

By interchanging R_1 and R_2 in the formula above, we obtain the second inequality. $\square$

Given a bounded complete Reinhardt domain R in $\mathbb{C}^n$, it is immediate to check that $S(R, tR) = t^{-1}$ and $S(tR, R) = t$ for all $t>0$, hence we get another elementary but useful consequence.

Corollary 2.6. *Let R be a bounded complete Reinhardt domain in $\mathbb{C}^n$ and $t>0$. Then we have $K(R) = K(tR)$.*

By Boas and Khavinson [10, Theorem 3] we know that $K(R) \geqslant \frac{1}{3\sqrt{n}}$ for each R. Then Lemma 2.5 combined with Aizenberg's result from (1.1) yields the following general lower bound.

Theorem 2.7. *Let R be a bounded complete Reinhardt domain in $\mathbb{C}^n$. Then, we have*

$$\max\left(\frac{1}{3\sqrt{n}}, \frac{1}{3\sqrt[3]{e}S(R, B_{\ell_1^n})S(B_{\ell_1^n}, R)}\right) \leqslant K(R).$$

Next, we give a general upper estimate for the first Bohr radius.

Theorem 2.8. *Let R be a bounded complete Reinhardt domain in $\mathbb{C}^n$. We have*

$$K(R) \leqslant e^3 2^{3/2}\sqrt{\log n}\,\frac{\sup_{z\in R}(\sum_{k=1}^n |z_k|^2)^{1/2}}{\sup_{z\in R}\sum_{k=1}^n |z_k|},$$

or equivalently

$$K(R) \leqslant e^3 2^{3/2}\sqrt{\log n}\,\frac{S(R, B_{\ell_2^n})}{S(R, B_{\ell_1^n})}.$$

Proof. By Proposition 2.2 and Corollary 2.4 we have that for each m

$$\begin{aligned}
K(R) &\leqslant K_m(R) \leqslant \left(\sqrt{\log n}\sqrt{m!}\,2^{\frac{3m-1}{2}} m^{\frac{3}{2}}\right)^{\frac{1}{m}} \left(\frac{\sup_{z\in R}(\sum_{k=1}^n |z_k|^2)^{1/2}}{\sup_{z\in R}\sum_{k=1}^n |z_k|}\right)^{\frac{m-1}{m}} \\
&= \left(\sqrt{\log n}\sqrt{m!}\,2^{\frac{3m-1}{2}} m^{\frac{3}{2}}\right)^{\frac{1}{m}} \left(\frac{\sup_{z\in R}\sum_{k=1}^n |z_k|}{\sup_{z\in R}(\sum_{k=1}^n |z_k|^2)^{1/2}}\right)^{\frac{1}{m}} \\
&\quad\times \frac{\sup_{z\in R}(\sum_{k=1}^n |z_k|^2)^{1/2}}{\sup_{z\in R}\sum_{k=1}^n |z_k|}.
\end{aligned}$$

Now, we consider $t>0$ such that tR is a subset of the polydisc $B_{\ell_\infty^n}$ and there exists $z_0 \in tR$ satisfying $||z_0||_\infty = 1$. We have

$$\left(\frac{\sup_{z\in R}\sum_{k=1}^n |z_k|}{\sup_{z\in R}(\sum_{k=1}^n |z_k|^2)^{1/2}}\right)^{\frac{1}{m}} = \left(\frac{\sup_{z\in tR}\sum_{k=1}^n |z_k|}{\sup_{z\in tR}(\sum_{k=1}^n |z_k|^2)^{1/2}}\right)^{\frac{1}{m}} \leqslant n^{1/m}. \tag{2.7}$$

Moreover, in the proof of Defant et al. [12, Theorem 4.2] it is shown that there exists an m ($m = 1$ if $n = 2$ and $m = [\log n]$ if $n \geqslant 3$) such that

$$\left(\sqrt{\log n}\sqrt{m!}2^{\frac{3m-1}{2}}m^{\frac{3}{2}}\right)^{\frac{1}{m}} n^{1/m} < e^3 2^{\frac{3}{2}}\sqrt{\log n} \tag{2.8}$$

from which the conclusion follows. □

We obtain the following interesting special case.

Corollary 2.9. *Let R be a bounded complete Reinhardt domain in $\mathbb{C}^n$ such that $B_{\ell_1^n} \subset R \subset B_{\ell_2^n}$. Then*

$$\frac{1}{3\sqrt[3]{e}}\frac{1}{\sup_{z\in R}\sum_{k=1}^n |z_k|} \leqslant K(R) \leqslant e^3 2^{3/2}\sqrt{\log n}\frac{1}{\sup_{z\in R}\sum_{k=1}^n |z_k|}.$$

Proof. It is immediate from our lower and upper estimates (Theorems 2.7 and 2.8) as $S(R, B_{\ell_2^n}) \leqslant 1$, $S(B_{\ell_1^n}, R) \geqslant 1$ and, by (2.5), $S(R, B_{\ell_1^n}) = \sup_{z\in R}\sum_{k=1}^n |z_k|$. □

Define for $0 < p_k \leqslant \infty$, $k = 1, \dots, n$, the bounded complete Reinhardt domain

$$R_{(p_k)} \coloneqq \{z \in \mathbb{C}^n : \sum_{k=1}^n |z_k|^{p_k} < 1\}.$$

Corollary 2.10. *Let $1 \leqslant p_k \leqslant 2$, $k = 1, \dots, n$. Then we have*

$$\frac{1}{3\sqrt[3]{e}}\frac{1}{\sup_{z\in R_{(p_k)}}\sum_{k=1}^n |z_k|} \leqslant K(R_{(p_k)}) \leqslant e^3 2^{3/2}\sqrt{\log n}\frac{1}{\sup_{z\in R_{(p_k)}}\sum_{k=1}^n |z_k|}.$$

Sometimes more precise estimates are possible:

Example 2.11. If $1 \leqslant p_k \leqslant 2$, $k = 1, \dots, 2n$ and $p_{2k} = 2$, $k = 1, \dots, n$, then

$$\frac{1}{3\sqrt{2n}} \leqslant K(R_{(p_k)}) \leqslant e^3 2^{3/2}\sqrt{\log 2n}\frac{1}{\sqrt{2n}}.$$

This result clearly follows from Theorem 2.7 and the fact $\sup_{z\in R_{(p_k)}}\sum_{k=1}^{2n} |z_k| \geqslant \sqrt{n}$.

We conjecture that there exists a constant $C > 0$ such that if $2 \leqslant p_k \leqslant \infty$, $k = 1, \dots, n$, then

$$\frac{1}{C}\frac{1}{\sqrt{n}} \leqslant K(R_{(p_k)}) \leqslant C\sqrt{\log n}\frac{1}{\sqrt{n}}.$$

3. General estimates for the second Bohr radius

The upper estimate for the second Bohr radius is again based on our probabilistic tool Theorem 2.3. Again, we begin with some elementary facts, which are analogues of Lemma 2.5 and its corollary.

Lemma 3.1. *Let R_1 and R_2 be two bounded complete Reinhardt domains in $\mathbb{C}^n$, and $t>0$. Then*

(1) $B(R_1) \leqslant S(R_1, R_2) S(R_2, R_1) B(R_2)$.
(2) $B(R_1) = B(tR_1)$.
(3) $B(R_1) \leqslant tB(R_2)$, *whenever* $R_2 \subset R_1 \subset tR_2$.

Proof. The proof of (1) follows the same pattern as the proof of Lemma 2.5. Also (2) is obtained from (1) in an analogous way as Corollary 2.6 is obtained from Lemma 2.5. Finally, (3) is a consequence of (1) since $S(R_2, R_1) \leqslant 1$ and $S(R_1, R_2) \leqslant t$. □

As $B(B_{\ell_\infty^n}) = K(B_{\ell_\infty^n})$ (see the introduction) and $\frac{1}{3\sqrt{n}} \leqslant K(B_{\ell_\infty^n})$ (see (1.1)), we get, using also (1.3), the following lower estimate of the second Bohr radius.

Proposition 3.2. *Let R be a bounded complete Reinhardt domain in $\mathbb{C}^n$. We have*

$$\frac{1}{3} \max\left(\frac{1}{n}, \frac{1}{\sqrt{n} S(R, B_{\ell_\infty^n}) S(B_{\ell_\infty^n}, R)}\right) \leqslant B(R).$$

In order to formulate our upper estimates we again need some more notation: For a bounded complete Reinhardt domain R in $\mathbb{C}^n$ we write

$$\begin{aligned} b_m(R) &:= \left(\inf_{|\alpha|=m} \sup_{z \in R} |z^\alpha|\right)^{-\frac{1}{m}}, \quad m \in \mathbb{N} \cup \{0\}, \\ a_2(R) &:= b_1(R), \\ a_n(R) &:= b_{[\log n]}(R), \quad n \geqslant 3. \end{aligned} \tag{3.1}$$

Clearly, if R_1 and R_2 are complete Reinhardt domains in $\mathbb{C}^n$ such that $R_1 \subset R_2$ then $a_n(R_2) \leqslant a_n(R_1)$.

Theorem 3.3. *For each bounded complete Reinhardt domain R in $\mathbb{C}^n$ we have*

$$B(R) \leqslant e^3 2^{3/2} \sqrt{\log n}\, a_n(R) \frac{\sup_{z \in R} (\sum_{k=1}^n |z_k|^2)^{1/2}}{n}.$$

Proof. By Theorem 2.3, we know that there are signs $\varepsilon_\alpha = \pm 1$, $|\alpha| = m$, such that

$$\sup_{z\in R}\left|\sum_{|\alpha|=m} \varepsilon_\alpha \frac{m!}{\alpha!} z^\alpha\right|$$

$$\leqslant \sqrt{\log n}\,2^{\frac{3}{2}m-\frac{1}{2}} m^{\frac{3}{2}} \sup_{|\alpha|=m} \sqrt{\frac{m!}{\alpha!}} \sup_{z\in R}\left(\sum_{k=1}^n |z_k|^2\right)^{\frac{m-1}{2}} \sup_{z\in R} \sum_{k=1}^n |z_k|$$

$$\leqslant \sqrt{\log n}\,2^{\frac{3}{2}m-\frac{1}{2}} m^{\frac{3}{2}} \sqrt{m!} \sup_{z\in R}\left(\sum_{k=1}^n |z_k|^2\right)^{\frac{m-1}{2}} \sup_{z\in R} \sum_{k=1}^n |z_k|.$$

Hence, by definition of the second Bohr radius, we get for every $0<\delta<1$

$$\sum_{|\alpha|=m} \frac{m!}{\alpha!} \sup_{z\in(1-\delta)B(R)R} |z^\alpha|$$

$$\leqslant \sqrt{\log n}\,2^{\frac{3}{2}m-\frac{1}{2}} m^{\frac{3}{2}} \sqrt{m!} \sup_{z\in R}\left(\sum_{k=1}^n |z_k|^2\right)^{\frac{m-1}{2}} \sup_{z\in R} \sum_{k=1}^n |z_k|.$$

Thus

$$n^m B(R)^m b_m(R)^{-m} \leqslant \sqrt{\log n}\,2^{\frac{3}{2}m-\frac{1}{2}} m^{\frac{3}{2}} \sqrt{m!} \sup_{z\in R}\left(\sum_{k=1}^n |z_k|^2\right)^{\frac{m-1}{2}} \sup_{z\in R} \sum_{k=1}^n |z_k|.$$

By taking the mth root

$$B(R) \leqslant \left(\sqrt{\log n}\sqrt{m!}\,2^{\frac{3m-1}{2}} m^{\frac{3}{2}}\right)^{\frac{1}{m}} \left(\frac{\sup_{z\in R}\sum_{k=1}^n |z_k|}{\sup_{z\in R}(\sum_{k=1}^n |z_k|^2)^{1/2}}\right)^{\frac{1}{m}} b_m(R) \times \frac{\sup_{z\in R}(\sum_{k=1}^n |z_k|^2)^{\frac{1}{2}}}{n}.$$

By (2.7) we get

$$B(R) \leqslant \left(\sqrt{\log n}\sqrt{m!}\,2^{\frac{3m-1}{2}} m^{\frac{3}{2}}\right)^{\frac{1}{m}} n^{1/m} b_m(R) \frac{\sup_{z\in R}(\sum_{k=1}^n |z_k|^2)^{\frac{1}{2}}}{n}.$$

Finally, by taking $m=1$ if $n=2$, $m=[\log n]$ if $n\geqslant 3$ and applying (2.8) we obtain

$$B(R) \leqslant e^3 2^{3/2} \sqrt{\log n}\, a_n(R) \frac{\sup_{z\in R}(\sum_{k=1}^n |z_k|^2)^{\frac{1}{2}}}{n}. \qquad \square$$

The following result combines the preceding two statements.

Corollary 3.4. *Let R be a bounded complete Reinhardt domain in $\mathbb{C}^n$. Then we have*

$$\frac{1}{3}\max\left\{\frac{1}{n},\frac{1}{\sqrt{n}S(R,B_{\ell_\infty^n})S(B_{\ell_\infty^n},R)}\right\}\leqslant B(R)\leqslant e^3 2^{3/2}\sqrt{\log n}\,a_n(R)\,\frac{S(R,B_{\ell_2^n})}{n}.$$

In particular, if $R\subset B_{\ell_2^n}$, then

$$\frac{1}{3n}\leqslant B(R)\leqslant e^3 2^{3/2}\sqrt{\log n}\,a_n(R)\,\frac{1}{n}.$$

Lemma 3.5. *Let $p_1,\dots,p_n>0$. Then for each multi-index $\alpha\in(\mathbb{N}\cup\{0\})^n$ we have*

$$\sup\{|z^\alpha| : z\in R_{(p_k)}\}=\frac{(\frac{\alpha_1}{p_1})^{\frac{\alpha_1}{p_1}}\dots(\frac{\alpha_n}{p_n})^{\frac{\alpha_n}{p_n}}}{(\frac{\alpha_1}{p_1}+\dots+\frac{\alpha_n}{p_n})^{\frac{\alpha_1}{p_1}+\dots+\frac{\alpha_n}{p_n}}}.$$

Proof. Given $\beta=(\beta_k)$, $u=(u_k)\in[0,+\infty)^n$ we denote $|\beta|=\beta_1+\dots+\beta_n$, $u^\beta=u_1^{\beta_1}\dots u_n^{\beta_n}$ and $Q=\{u\in[0,+\infty)^n : u_1+\dots+u_n\leqslant 1\}$. Clearly, it is enough to prove that

$$\sup\{u^\beta : u\in Q\}=\frac{\beta^\beta}{|\beta|^{|\beta|}}\quad\text{for all }\beta\in[0,+\infty)^n.$$

To do that we need the following inequality (concavity of log): *Given $a_1,\dots,a_n>0$ and $q_1,\dots,q_n\geqslant 1$ such that $\frac{1}{q_1}+\dots+\frac{1}{q_n}=1$, it holds that $a_1\dots a_n\leqslant\frac{a_1^{q_1}}{q_1}+\dots+\frac{a_n^{q_n}}{q_n}$.*

We can assume that $\beta_k>0$ for all $k=1,\dots,n$. Let $u\in Q$. By applying the concavity of log with $q_k:=\frac{|\beta|}{\beta_k}$ and $a_k:=(\frac{u_k}{\beta_k})^{\frac{\beta_k}{|\beta|}}$ for all $k=1,\dots,n$,

$$\left(\frac{u^\beta}{\beta^\beta}\right)^{\frac{1}{|\beta|}}\leqslant\frac{\beta_1}{|\beta|}\frac{u_1}{\beta_1}+\dots+\frac{\beta_n}{|\beta|}\frac{u_n}{\beta_n}=\frac{u_1+\dots+u_n}{|\beta|}.$$

Hence

$$u^\beta\leqslant\frac{\beta^\beta}{|\beta|^{|\beta|}}$$

for all $u\in Q$. But, if we take $u=(u_k)_{k=1}^n$ with $u_k:=\frac{\beta_k}{|\beta|}$ for all $k=1,\dots,n$, we have that $u_1+\dots+u_n=1$ and $u^\beta=\frac{\beta^\beta}{|\beta|^{|\beta|}}$. This completes the proof. $\square$

For $z=((\frac{\alpha_1}{|\alpha|})^{\frac{1}{p_1}},\dots,(\frac{\alpha_n}{|\alpha|})^{\frac{1}{p_n}})$, we obtain

$$\sup\{|z^\alpha| : z\in R_{(p_k)}\}\geqslant\frac{\alpha_1^{\frac{\alpha_1}{p_1}}\dots\alpha_n^{\frac{\alpha_n}{p_n}}}{|\alpha|^{\frac{\alpha_1}{p_1}+\dots+\frac{\alpha_n}{p_n}}}.\tag{3.2}$$

If in the above lemma we take $p_1 = \cdots = p_n = p \geqslant 1$, then $\sup_{z \in \ell_p^n} |z^\alpha| = (\frac{\alpha^\alpha}{|\alpha|^{|\alpha|}})^{1/p}$, a result which can be found e.g. in [15, Lemma 1.38].

Example 3.6. Let $n > 2$ and $p_1, \ldots, p_n > 0$.

(1) If all $0 < p_k \leqslant 2$, then
$$\frac{1}{3n} \leqslant B(R_{(p_k)}) \leqslant e^3 2^{3/2} (\log n)^{\frac{1}{2} + 1/\min p_k} \frac{1}{n}.$$

(2) If all $2 \leqslant p_k < \infty$, then
$$\frac{1}{3n^{\frac{1}{2}+1/\min p_k}} \leqslant B(R_{(p_k)}) \leqslant e^3 2^{3/2} (\log n)^{\frac{1}{2}+1/\min p_k} \frac{1}{n^{\frac{1}{2}+1/\max p_k}}.$$

Proof. Define $p := \min p_k$, and get, by (3.2) for all m
$$\sup\{|z^\alpha| : z \in R_{(p_k)}\} \geqslant \frac{\alpha_1^{\frac{\alpha_1}{p_1}} \ldots \alpha_n^{\frac{\alpha_n}{p_n}}}{m^{\frac{\alpha_1}{p_1} + \cdots + \frac{\alpha_n}{p_n}}} \geqslant \frac{1}{m^{\frac{m}{p}}}$$
for all $\alpha \in (\mathbb{N} \cup \{0\})^n$ such that $|\alpha| = m$. Hence for all m
$$\inf_{|\alpha|=m} \sup\{|z^\alpha| : z \in R_{(p_k)}\} \geqslant \frac{1}{m^{\frac{m}{p}}}$$
and therefore
$$b_m(R_{(p_k)}) = \left(\frac{1}{\inf_{|\alpha|=m} \sup\{|z^\alpha| : z \in R_{(p_k)}\}} \right)^{\frac{1}{m}} \leqslant m^{\frac{1}{p}}.$$
Thus
$$a_n(R_{(p_k)}) = b_{[\log n]}(R_{(p_k)}) \leqslant (\log n)^{\frac{1}{\min p_k}}. \tag{3.3}$$
Now, if we assume $0 < p_k \leqslant 2$, $k = 1, \ldots, n$, then $S(R_{(p_k)}, B_{\ell_2^n}) \leqslant 1$. Applying Corollary 3.4, we obtain (1). In the case that $2 \leqslant p_k < \infty$ for all $k = 1, \ldots, n$, we have $S(R_{(p_k)}, B_{\ell_\infty^n}) \leqslant 1$ and $S(B_{\ell_\infty^n}, R_{(p_k)}) = b$ such that
$$\frac{1}{b^{p_1}} + \cdots + \frac{1}{b^{p_n}} = 1.$$
To get an upper estimate of b we observe that $B_{\ell_p^n} \subset R_{(p_k)}$. Hence, by (2.6)
$$S(R_{(p_k)}, B_{\ell_\infty^n}) S(B_{\ell_\infty^n}, R_{(p_k)}) \leqslant S(B_{\ell_\infty^n}, B_{\ell_p^n}) = n^{\frac{1}{p}}.$$
If we denote $q := \max p_k$, we have $R_{(p_k)} \subset B_{\ell_q^n}$, then by (2.6)
$$S(R_{(p_k)}, B_{\ell_2^n}) \leqslant S(B_{\ell_q^n}, B_{\ell_2^n}) = n^{\frac{1}{2} - \frac{1}{q}}.$$
Finally, (3.3) and Corollary 3.4 give (2). $\square$

Note that if in the above example, we consider the case that all $0 < p_k = p$, then we recover (1.3), the asymptotic estimates for $B(B_{\ell_p^n})$ obtained by Boas for $1 \leqslant p \leqslant \infty$ in [9, Theorem 5] (but our log-term is worse whenever $1 \leqslant p < 2$). In any case the asymptotic estimates obtained in this example are far from sharp as the next example shows.

Example 3.7. Fix $r \in \mathbb{N}$. Given $p_1, \dots, p_r \geqslant 2$, let $R := \{z \in \mathbb{C}^n : |z_1|^{p_1} + \cdots + |z_r|^{p_r} + |z_{r+1}|^2 + \cdots + |z_n|^2 < 1\}$. Then, we have

$$\frac{1}{3n} \leqslant B(R) \leqslant C \frac{\log n}{n},$$

for all $n > r$, where $0 < C \leqslant e^3 2^{3/2} (r+1)^{1/2}$.

The left inequality is obtained in Example 3.6. On the other hand it is very easy to check $S(R, B_{\ell_2^n}) \leqslant (r+1)^{1/2}$.

Let us collect some more examples in order to illustrate our results. Recall the definition of mixed Minkowski spaces:

$$\ell_p^m(\ell_q^n) := \{(x_k)_{k=1}^m | x_1, \dots, x_m \in \mathbb{C}^n\} \text{ with } \|(x_k)_{k=1}^m\|_{p,q} := \left(\sum_{k=1}^m \|x_k\|_q^p \right)^{1/p}.$$

Example 3.8. Let $m, n \in \mathbb{N}$ with $m, n \geqslant 2$.

(1) For $0 < p, q \leqslant 2$

$$\frac{1}{3} \frac{1}{mn} \leqslant B(B_{\ell_p^m(\ell_q^n)}) \leqslant e^3 2^{3/2} (\log mn)^{\frac{1}{2} + \frac{1}{\min\{p,q\}}} \frac{1}{mn}.$$

(2) For $2 \leqslant p, q \leqslant \infty$

$$\frac{1}{3} \frac{1}{m^{\frac{1}{2}+\frac{1}{p}} n^{\frac{1}{2}+\frac{1}{q}}} \leqslant B(B_{\ell_p^m(\ell_q^n)}) \leqslant e^3 2^{3/2} (\log mn)^{\frac{1}{2} + \frac{1}{\min\{p,q\}}} \frac{1}{m^{\frac{1}{2}+\frac{1}{p}} n^{\frac{1}{2}+\frac{1}{q}}}.$$

Proof. The space $\ell_p^m(\ell_q^n)$ has dimension mn. By applying Corollary 3.4

$$\begin{aligned} \frac{1}{3} \max \left\{ \frac{1}{mn}, \frac{1}{\sqrt{mn} S(B_{\ell_p^m(\ell_q^n)}, B_{\ell_\infty^{mn}}) S(B_{\ell_\infty^{mn}}, B_{\ell_p^m(\ell_q^n)})} \right\} &\leqslant B(B_{\ell_p^m(\ell_q^n)}) \\ &\leqslant e^3 2^{3/2} \sqrt{\log mn}\ a_{mn}(B_{\ell_p^m(\ell_q^n)}) \frac{S(B_{\ell_p^m(\ell_q^n)}, B_{\ell_2^{mn}})}{mn}. \end{aligned} \tag{3.4}$$

As $B_{\ell_{\min\{p,q\}}^{mn}} = B_{\ell_{\min\{p,q\}}^m(\ell_{\min\{p,q\}}^n)} \subset B_{\ell_p^m(\ell_q^n)}$, by (3.3) we get that for all $p, q > 0$

$$a_{mn}(B_{\ell_p^m(\ell_q^n)}) \leqslant (\log mn)^{\frac{1}{\min\{p,q\}}}.$$

Moreover, if $0 < p, q \leqslant 2$ then $S(B_{\ell_p^m(\ell_q^n)}, B_{\ell_2^{mn}}) \leqslant 1$ and we obtain (1).

The inclusion $B_{\ell_p^m(\ell_q^n)} \subset B_{\ell_\infty^{mn}}$ implies $S(B_{\ell_p^m(\ell_q^n)}, B_{\ell_\infty^{mn}}) \leqslant 1$. In the proof of Defant et al. [12, Example 4.5] it is pointed out that for $1 \leqslant a, b, c, d \leqslant \infty$

$$S(B_{\ell_a^m(\ell_b^n)}, B_{\ell_c^m(\ell_d^n)}) = S(B_{\ell_a^m}, B_{\ell_c^m}) S(B_{\ell_b^n}, B_{\ell_d^n}),$$

hence by (2.6)

$$S(B_{\ell_\infty^{mn}}, B_{\ell_p^m(\ell_q^n)}) = S(B_{\ell_\infty^m}, B_{\ell_p^m}) S(B_{\ell_\infty^n}, B_{\ell_q^n}) = m^{\frac{1}{p}} n^{\frac{1}{q}},$$

$$S(B_{\ell_p^m(\ell_q^n)}, B_{\ell_2^{mn}}) = S(B_{\ell_p^m}, B_{\ell_2^m}) S(B_{\ell_q^n}, B_{\ell_2^n}) = m^{\frac{1}{2}-\frac{1}{p}} n^{\frac{1}{2}-\frac{1}{q}}$$

for all $2 \leqslant p, \ q \leqslant \infty$. Now (2) follows from (3.4). $\square$

Remark 3.9. There exists an one to one correspondence between bounded convex complete Reinhardt domains in $\mathbb{C}^n$ and the open unit balls of the norms in $\mathbb{C}^n$ for which $(e_k)_{k=1}^n$, the canonical basis of $\mathbb{C}^n$, is 1-unconditional. Indeed, the Minkowski gauge of R (i.e., $||z||_R := \inf\{\lambda : z \in \lambda R\}, \ z \in \mathbb{C}^n$) is a norm on $\mathbb{C}^n$ and R coincides with the open unit ball of $(\mathbb{C}^n, ||.||_R)$. The fact that R is n-circular and balanced implies that $(e_k)_{k=1}^n$ is an 1-unconditional basis of $(\mathbb{C}^n, ||.||_R)$. Reciprocally, if $X = (\mathbb{C}^n, ||.||)$ is a Banach space, such that $(e_k)_{k=1}^n$ is an 1-unconditional basis, then it is trivial that its open unit ball B_X is a bounded convex complete Reinhardt domain in $\mathbb{C}^n$. Hence, the study done in [12] about the first Bohr radii for the unit balls of finite dimensional complex Banach spaces $X = (\mathbb{C}^n, ||.||)$ for which the canonical bases are normalized and 1-unconditional, is essentially the study of convex bounded complete Reinhardt domains in $\mathbb{C}^n$. But there it is very useful to be able to apply many results from the local Banach space theory. For the second Bohr radius a parallel study could be made. We are going to state only two corollaries to illustrate this possible development. We refer to [12,19] for the unexplained terms.

Corollary 3.10. *Let $X = (\mathbb{C}^n, ||.||)$ be a 2-convex symmetric Banach space such that $M^{(2)}(X) = 1$. Then for all $n \geqslant 2$*

$$\frac{1}{3} \frac{\sup_{||z||<1} (\sum_{k=1}^n |z_k|^2)^{\frac{1}{2}}}{n} \leqslant B(B_X) \leqslant e^3 2^{3/2} \sqrt{\log n}\, a_n(B_X) \frac{\sup_{||z||<1} (\sum_{k=1}^n |z_k|^2)^{\frac{1}{2}}}{n}.$$

Proof. The upper bound is a direct consequence of Theorem 3.3. For the lower bound we apply Proposition 3.2 to get

$$\frac{1}{3\sqrt{n} S(B_X, B_{\ell_\infty^n}) S(B_{\ell_\infty^n}, B_X)} \leqslant B(B_X).$$

Since, by hypothesis, $B_X \subset B_{\ell_\infty^n}$, we have $S(B_X, \ell_\infty^n) \leqslant 1$. Now, to finish the proof it is enough to show that

$$\frac{\sup_{||z||<1} (\sum_{k=1}^n |z_k|^2)^{\frac{1}{2}}}{n} = \frac{1}{\sqrt{n} S(B_{\ell_\infty^n}, B_X)}.$$

To prove this equality we follow the pattern of Defant et al. [12, Corollaries 5.3, 5.4]. The canonical basis is 1-unconditional, thus

$$S(B_{\ell_\infty^n}, B_X) = \sup\left\{ \left\| \sum_{k=1}^n z_k e_k \right\| : |z_k| \leqslant 1 \ k = 1, \dots, n \right\} = \left\| \sum_{k=1}^n e_k \right\|.$$

Moreover, since X is symmetric, by Lindenstrauss and Tzafriri [19, Proposition 3.a.6], we have that

$$\left\| \sum_{k=1}^n e_k^* \right\|_{X^*} \left\| \sum_{k=1}^n e_k \right\|_X = n.$$

Finally, in [14, Proposition 3.5] (see also [21, Proposition 3.5]) it is shown that under our assumptions on X

$$\sup_{||z||<1} \left(\sum_{k=1}^n |z_k|^2 \right)^{\frac{1}{2}} = \frac{||\sum_{k=1}^n e_k^*||_{X^*}}{\sqrt{n}}. \qquad \square$$

The last corollary is an analogue for the second Bohr radius of the one given for the first Bohr radius in [12, Corollary 5.4]. The proof follows a similar pattern.

Corollary 3.11. *Let $X = (\mathbb{C}^n, ||.||)$ be a symmetric Banach space. Then*

$$B(B_X)B(B_{X^*}) \leqslant C^2 \log n \frac{d(X, \ell_2^n)}{n}.$$

Moreover, if we assume $M^{(2)}(X) = 1$ then

$$\frac{1}{C^2} \frac{d(X, \ell_2^n)}{n} \leqslant K(B_X)K(B_{X^*}) \leqslant C^2 \log n \frac{d(X, \ell_2^n)}{n}$$

for all $n \geqslant 2$, where $0 < C \leqslant e^3 2^{3/2}$.

Since $d(X, \ell_2^n) \leqslant \sqrt{n}$ (see e.g. [22, p. 249]) we obtain that for any sequence (X_n) of symmetric n-dimensional Banach spaces $X_n = (\mathbb{C}^n, ||.||_n)$

$$\lim_n B(B_{X_n})B(B_{X_n^*}) = 0. \tag{3.5}$$

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